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Coupled Layerwise Theories for Hybrid and Sandwich Piezoelectric Beams

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Abstract— This paper presents review of the available one dimensional (1D) models of hybrid and sandwich beams and highlights the need of computationally efficient and accurate electromechanical coupled 1D beam models. The paper covers the discussion of uncoupled equivalent single layer theories, coupled equivalent single layer theories, layerwise theories, efficeint layerwise theories, 3D theories, efficient coupled zigzag theories, finite element models, exact piezoelasticity solution and coupled third order smeared beam models.

Index Terms— electromechancal loads, finite element, hybrid beams, laminate theories, piezoelectric, sandwich beams, zigzag models

1 INTRODUCTION

COMPOSITE laminates and sandwich structures with some embedded or surface bonded piezoelectric layers form part of a new generation of smart adaptive structures which have received enormous research attention in recent years. The sensing and actuation capability of piezoelectric layer is used for achieving active vibration control, shape control, noise control, health monitoring etc. Sandwich structures have high ratio of flexural stiffness to weight ratio resulting in lower deflection, higher buckling load and higher natural frequencies compared to other configuration of this type of structures. Sandwich structures offer advantage of placement of electrodes for the piezoelectric layers. Due to inhomogeneity in the mechanical properties across the thickness and presence of electric heterogeneity caused by the embedded piezoelectric layers, these structures can be analysed accurately and efficiently by coupled electromechanical three dimensional (3D), two dimensional (2D) and one dimensional (1D) models for solids, plates and shells, and beams, respectively.

The relevant literature is reviewed from the view point of this objective. Beginning with the pioneering book by Tiersten [1] on piezoelectric plate vibration, several books [2], [3] have been written on piezoelectric and smart structures. Several articles [4], [5] have surveyed on the research of hybrid composite and sandwich piezoelectric structures.

2 CLASSIFICATION OF THEORIES

The majority of the reported piezoelectric laminate theories fall into following categories. Their points of differentiation are based on assumed displacement field and the exclusion or inclusion of the piezoelectric coupling by independent electric potential variables.

2.1 Uncoupled Equivalent Single Layer Theories

In single layer beam or shell, the displacement field is assumed to have the same functional dependence on the thickness coordinate z for all layers. The deflection w is usually assumed to be independent of z. The electric potential is not included as an independent state variable and the piezoelectric coupling is neglected. The electric potential across each actuated piezoelectric layer is assumed to vary linearly across the thickness and induces piezoelectric strains. The electric field in the sensory layer can be back calculated from the mechanical strains using constitutive equations. Many reported theories belong to this class. In the classical laminate theory (CLT), the transverse shear and normal strains are neglected and so the axial displacement u and deflection w are approximated as

$$\begin{aligned} u(x,z,t) &= u_0(x,z,t) - z w_{0,x} \\ w &= w_0(x,t) \end{aligned}$$

In the first order shear deformation theory (FSDT), uniform transverse shear strain across the thickness is assumed and so the displacements are approximated in terms of three displacement variables u_0, ψ_0, and w_0 as:

$$\begin{aligned} u(x,z,t) &= u_0(x,z,t) + z\varphi_0(x,t) \\ w &= w_0(x,t) \end{aligned}$$

Since the actual shear strain distribution is not uniform, a shear correction factor is introduced to enhance this. No shear correction factor is required in higher order theories in which shear deformation is nonuniform across the thickness with u being a function of z of degree higher than in FSDT. In third order theory, u is taken as cubic in z such that the shear strain γ_{zx} is zero at the top and bottom of the beam of thickness h and displacement field is expressed in terms of u_0, ψ_0, w_0 as:

$$\begin{aligned} u(x,z,t) &= u_0(x,z,t) - z w_{0,x} + z\left(1 - \frac{3z^2}{h^2}\right)\varphi_0(x,t) \\ w &= w_0(x,t) \end{aligned}$$

2.2 Coupled Equivalent Single Layer Theories

The displacement field is assumed to have the same functional dependence on z for all layers and electric potentials are treated as independent state variables with layerwise approximation of electric potential φ across the thickness. Usually w is assumed to be independent of z and φ is assumed to be layerwise linear. The piezoelec-

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tric coupling is explicitly considered through direct and converse piezoelectric effects. Coupled CLT, coupled FSDT and coupled third order theory belong to this class.

2.3 Layerwise Theories

The axial displacement field is approximated across the thickness, layerwise or sublaminate (a set of layers) wise with continuity at each interface, and w is usually assumed to be independent of z. In the uncoupled layerwise theories, the electric potentials are not independent state variables. In the coupled layerwise theories, the electric potentials are independent state variables and φ is approximated across the thickness layerwise, sublayerwise or sublaminate (a set of layers) wise, and u and φ are usually assumed piecewise linear across the thickness. These theories attempt to reproduce the performance of 3D theories with less analytical complexity and computational effort and yield accurate values of intralaminar stresses and interlaminar shear stresses for thin and thick piezoelectric laminates and can handle arbitrary electrical configurations. Consequently, they offer analytical flexibility and robustness at an increased computatinal effort since the number of primary displacement variables depends on the number of layers. The computational cost is almost one order more than that of equivalent single layer theories. The shear stress τ_{zx} computed from constitutive equations, for theories in categories 1,2,3 is generally not continuous at the layer interfaces.

2.4 Efficient Layerwise Theories

The assumptions of displacement and electric potential fields are the same as in layerwise theories above with additional quadratic and cubic global dependence of u on z. The number of primary displacement variables is reduced to three by enforcing the shear traction free conditions at top and bottom and the conditions of continuity of τ_{zx} at the layer interfaces. These theories are computationally efficient as the number of primary displacement variables is only three and this does not depend on the number of layers. In the theories described in categories 2.1 to 2.4, τ_{zx} and σ_x predicted from constitutive equations can be improved upon (corrected) by integrating the equations of motion of 3D elasticity.

2.5 3D Theories

The 3D theories are based on assumed 3D expansions for all displacement components and electric potential which are made layerwise, sublayerwise or sublaminate wise. The 3D constitutive equations of a piezoelectric continuum and equations of motion and charge balance are used in the differential form or in the weak integral form. These theories directly yield interlaminar normal strain and stresses using the constitutive equations. Most importantly, these theories include the effect of the d_{33} piezoelectric coefficient which is neglected by most previous categories. The 3D constitutive equations of a piezoelectric orthotropic material of class mm2 symmetry, with principal material axes x_1, x_2, x_3 and polarised along direction x_3, with stress σ, engineering strain ϵ, electric field E and electric displacement D with respect to the principal material axes, are given by

$$\varepsilon = S\sigma + d^T E + \alpha\theta$$
$$D = d\sigma + \bar{\varepsilon}E + q\theta$$

where the superscript T denotes matrix transpose and θ is the temperature rise above the stress-free reference temperature and

$$S = \begin{bmatrix} s_{11} & s_{12} & s_{13} & 0 & 0 & 0 \\ s_{21} & s_{22} & s_{23} & 0 & 0 & 0 \\ s_{31} & s_{32} & s_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & s_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & s_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & s_{66} \end{bmatrix},$$

$$d^T = \begin{bmatrix} 0 & 0 & d_{31} \\ 0 & 0 & d_{32} \\ 0 & 0 & d_{33} \\ 0 & d_{24} & 0 \\ d_{15} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \alpha = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

$$\bar{\varepsilon} = \begin{bmatrix} \bar{\varepsilon}_{11} & 0 & 0 \\ 0 & \bar{\varepsilon}_{22} & 0 \\ 0 & 0 & \bar{\varepsilon}_{33} \end{bmatrix}, \qquad q = \begin{bmatrix} 0 \\ 0 \\ q_3 \end{bmatrix}.$$

2.6 Exact Piezothermoelasticity Solutions

These are exact solutions of the partial differential equations of equilibrium or motion and electric displacement conservation through the laminate thickness and exactly satisfy the continuity conditions at the layer interfaces and the boundary conditions at the boundary. These are exact 3D solutions for a laminated piezoelectricplate or shell, and exact 2D solutions for plane strain problem of cylindrical bending of a flat or curved panel or assumed generalised plane stress problem of a beam of small width. These solutions do not involve any approximations. Such solutions exist only for few specific geometric configurations, electromechanical boundary conditions and loadings. These are mostly used as benchmark tests for assessing other theories of laminates. Naturally, some of the reported formulations may not clearly fall in one of the above categories.

2.7 Coupled Zigzag Models

An efficient new coupled 1D theory was developed for piezoelectric hybrid beams under electromechanical static and dynamic load by extending the theory [31] for static analysis to a theory for any layup including piezoelectric layers of different materials [36], [37]. The model combines third order zigzag approximation for the displacement with layerwise approximation of the electric potential field as piecewise linear for sublayers. The transverse displacement is approximated to account for the piezoelectric transverse normal strain induced by the electric potential. By enforcing approximately (by neglecting the explicit contribution of φ to τ_{zx} the conditions of zero transverse shear stress at the top and bottom and its continuity at

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layer interfaces, the displacement field is expressed in terms of three primary displacement variables and potentials. This layerwise theory for displacement and potential fields thus preserves the computational advantage of an equivalent single layer (ESL) theory. The governing coupled equations of motion and charge balance and variationally consistent boundary conditions are derived from Hamilton's principle. Analytical Fourier series solutions are obtained, for static response, natural frequencies and forced response under harmonic load, for simply supported hybrid beams.

The theory is assessed by comparison of the results with the exact 2D piezoelastic solution and uncoupled FSDT solution. The present results are generally much more accurate than the FSDT solution and agree well with the exact solution for thin and moderately thick hybrid beams. The capability of the developed theory to adequately model open and closed circuit electric boundary conditions to accurately predict their influence on the response is demonstrated. The effects of ratio of span-to-thickness and ratio of piezolayer thickness to beam thickness on the response are investigated.

Another efficient new coupled 1D zigzag theory [38] was developed for static and dynamic analysis of hybrid beams under thermoelectromechanical loads, by extending the above zigzag model by including the explicit contribution of electric potential φ and temperature Θ in the conditions imposed on τ_{zx}. The axial displacement is approximated as a combination of global third order variation across the thickness with additional layerwise piecewise linear variation. The thermal and potential fields are approximated sublayer-wise as piecewise linear. The model considers both the axial and transverse electric fields. The deflection field is approximated to account for the transverse normal strain induced due to the piezoelectric d_{33} coefficient and the thermal expansion coefficient α_3. The displacement field is expressed in terms of only three primary displacement variables, electric potential variables and thermal field by satisfying exactly the conditions of zero transverse shear stress at the top and bottom and its continuity at layer interfaces. The governing coupled dynamic equations of stress and charge and variationally consistent boundary conditions are derived using Hamilton's principle.

The developed theory can accurately model open and closed circuit boundary conditions. The number of primary displacement unknowns is three, which is independent of the number of layers and equal in number to the ones used in the FSDT. This layerwise theory for displacement and potential fields thus preserves the computational advantage of an equivalent single layer (ESL) theory. Analytical Fourier series solutions are obtained for simply supported hybrid beams, for static response under electrothermomechanical load, natural frequencies of free vibrations and steady state undamped and damped forced response under harmonic load.

3 VARIOUS BEAM MODELS

A review of 3D continuum-based approaches, 2D theories for plates and shells and 1D theories for beams, along with their comparative study for plates under static loading, is presented by Saravanos and Heyliger [4]. In these papers, analytical 3D solutions are available only for some specific shapes and boundary conditions [6], [7] such as simply-supported infinite flat panels. The 3D finite element analysis for piezoelectric plates [8], [9] and 2D finite elements for beams result in large problem size which may become computationally costly for practical dynamics and control problems. Hence efficient accurate electrothermomechanical coupled 2D plate and 1D beam theories are required without too much loss of accuracy compared to 3D models.

Control of beam vibration using piezoelectric materials vibration was demonstrated by early researchers [10], [11].

Several beam theories of varying accuracy have been developed. Early works used elastic beam models [12], [13] with effective forces and moments due to induced strain of piezoelectric actuators. Finite element formulations were also presented [14], [15]. A discrete layer theory with layerwise approximation of displacements was developed for elastic laminated beams with induced actuation strain by Robins and Reddy [16]. Classical laminate theory (CLT) [17], [18] first order shear deformation theory (FSDT) [18], [19] and the refined third order theory (TOT) [20, 21] based on Reddy's theory [22], have been applied without electromechanical coupling to hybrid beams and plates. No shear correction factor is needed in third order theories.

Coupled CLT, FSDT [23], [24] and TOT [25], [26] for hybrid beams and plates including the charge equation of electrostatics and electromechanical coupling have been reported with layerwise linear approximation for the potential field. In these third order theories, the transverse shear strain γ_{zx} is zero at the top and bottom of the beam, but the shear traction free conditions at the top and bottom of the beam are not exactly satisfied. Saravanos and Heyliger [27] have presented coupled discrete layer theory (DLT), using layerwise approximation for displacements and electric potential, which yields accurate results for thin and thick beams. But it is expensive for practical problems since the number of displacement unknowns depend on the number of sublayers. Carrera [28] has presented a coupled DLT for plates with layerwise linear zig-zag approximation for axial displacement and quadratic one for transverse shear stresses and potential. But the axial electric field is neglected and the constitutive equation for shear stresses is only approximately satisfied. Several of these theories have been applied in the above works and in [29], [30] for control of vibration of hybrid beams with piezoelectric layers by active or passive damping strategy and also by hybrid active-passive intelligent constrained layer damping treatments.

Except for the coupled DLT [27], in which the transverse displacement is also taken as piecewise linear, no other 2D theory discussed above considers the piezoelectric transverse normal strain induced due to piezoelectricity through d_{33} coefficient, which has been observed to have considerable effect on the response, especially for electrical load [4]. To overcome the disadvantage of large number of displacement unknowns, dependent on the number of layers, in the DLT of Ref.[27], Kapuria [31] has recently developed a novel efficient coupled layerwise theory (DLT), for static analysis of hybrid beams, using a third order zigzag approximation for the axial displacement [32-34] with a sublayerwise piecewise linear approximation for the potential φ. The transverse displacement is approximated to account for the piezoelectric transverse normal strain induced by the electric potential through the piezoelectric strain constant d_{33}. The model considers both the axial and transverse electric fields. By neglecting the explicit contribution of φ, the conditions of zero transverse shear stress τ_{zx} at the top and bottom surfaces and the conditions of continuity of τ_{zx} at layer interfaces are enforced to formulate the theory in terms of only three displacement unknowns, which are independent of the number of layers and equal in number to the ones used in the FSDT. This DLT has the computational advantage of an equivalent single layer (ESL) theory and yet yields accurate through-the-thickness variations of displacements, electric

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field and stresses. Vidoli and Batra [35] have recently derived plate and rod equations for piezoelectric body from a mixed three dimensional variational principle.

4 Conclusion

A review is presented for different computational models for smart beams. Special emphasis is laid upon the development of different theories related to smart beams. Due to the inhomogeneity in the mechanical properties across the thickness and the presence of electric heterogeneity caused by the piezoelectric layer, the classical and first order shear deformation theories are inadequate for the analysis of thick and moderately thick beams and so third order and several higher order theories have been developed. This paper reviews different theories with latest development for smart beams starting with elastic beam models. Several beam theories of varying accuracy have been developed. Early works used elastic beam models with effective forces and moments due to the induced strain of the piezoelectric actuators. Finite element formulations were also presented.

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